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Mathematical Modeling of Shoreline Evolution

by
Bernard Le Mehaute and Mills Soldate

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A critical literature survey on mathematical modeling of shoreline evolution is presented. The emphasis is on long-term evolution rather than seasonal or evolution taking place during a storm. The one-line theory of Pelnard-Considere (1956) is developed along with a number of applications. Refinements to the theory are introduced by considering changes of beach slope, wave diffraction effects, wave variation, and variation of sea level. The case of hooked bays is also reviewed.

(continued)

It is concluded that a finite-difference mathematical scheme could be developed for engineering purposes for a small wave angle. For the large wave angle, shoreline instability does not permit use of a reliable mathematical model at this time.

PREFACE

This report is published to provide coastal engineers with a literature survey on mathematical modeling of shoreline evolution, which it is hoped will lead the way in establishing a flexible and practical numerical method suitable for predicting shoreline evolution resulting from the construction of navigation and shore structures. The work was carried out under the coastal structures program of the Coastal Engineering Research Center (CERC).

The report was prepared by Bernard Le Mehaute, senior vice president, and Mills Soldate, Tetra Tech, Inc., Pasadena, California, under CERC Contract No. DACW72-7T-C-0002. Funds for the preparation of this literature review part of the contract were provided by the U.S. Army Engineer Division, North Central, Chicago, Illinois.

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Dr. Weggel was the CERC contract monitor for the report, under the general supervision of G.M. Watts, Chief, Engineering Development Division.

Comments on this publication are invited.

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JOHN H. COUSINS
Colonel, Corps of Engineers
Commander and Director

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CONVERSION FACTORS, U.S. CUSTOMARY TO METRIC (SI)
UNITS OF MEASUREMENT

U.S. customary units of measurement used in this report can be converted to metric (SI) units as follows:

Multiply	by	To obtain
inches	25.4	millimeters
	2.54	centimeters
square inches	6.452	square centimeters
cubic inches	16.39	cubic centimeters
feet	30.48	centimeters
	0.3048	meters
square feet	0.0929	square meters
cubic feet	0.0283	cubic meters
yards	0.9144	meters
square yards	0.836	square meters
cubic yards	0.7646	cubic meters
miles	1.6093	kilometers
square miles	259.0	hectares
knots	1.8532	kilometers per hour
acres	0.4047	hectares
foot-pounds	1.3558	newton meters
millibars	1.0197×10^{-3}	kilograms per square centimeter
ounces	28.35	grams
pounds	453.6	grams
	0.4536	kilograms
ton, long	1.0160	metric tons
ton, short	0.9072	metric tons
degrees (angle)	0.1745	radians
Fahrenheit degrees	5/9	Celsius degrees or Kelvins[1]

[1]To obtain Celsius (C) temperature readings from Fahrenheit (F) readings, use formula: $C = (5/9)(F - 32)$.

To obtain Kelvin (K) readings, use formula: $K = (5/9)(F - 32) + 273.15$.

SYMBOLS AND DEFINITIONS

t	time
ox	horizontal axis at S W L parallel to the (initial) beach profile
oy	horizontal axis at S W L perpendicular to the (initial) beach profile
D	beach depth (depth beyond which sediment transport is negligible)
α	wave angle with beach profile
α_o	wave angle with beach profile at infinity
Q	longshore transport (littoral drift) discharge
K	constant $= \frac{1}{D} \frac{dQ}{d\alpha} \Big\vert_{\alpha = \alpha_o}$
u	parameter $u = \frac{x}{(4Kt)^{1/2}}$
$E(u)$	Fresnel integral $= E(u) = \frac{2}{\sqrt{u}} \int_u^\infty e^{-u^2} du$
ℓ	length of groin
t_1	time for the beach profile to reach the end of the groin
t_1'	transform time $t_1' = 0.62 t_1$
B	sinusoidal beach amplitude (at time $t = o$)

λ parameter related to beach wavelength $L{:}\lambda = \left(\frac{2\pi}{L}\right)^2 K$

X parametric value of x defining volume of beach dumping

Y parametric value of y defining volume of beach dumping

R parameter used to define hypocycloid beach profile between headlands

k_1

and coefficients used in the littoral drift formula to characterize the effect of wave angle

k_2

H_b breaking wave height

d_b water depth at inception of wave breaking

c_g group velocity

k littoral drift constant 6.42×10^{-3}

y_1 distance of shoreline from a horizontal axis parallel to the initial beach profile

y_2' distance of the offshore beach limit from a horizontal axis parallel to the initial beach profile

w equilibrium distance $y_2 - y_1$

y_2 $y_2' - w$

Q_y onshore-offshore transport per unit length of beach

q_y onshore-offshore transport parameter (dimension LT^{-1})

Q_1 longshore sand transport discharge in shallow water

Q_2 longshore sand transport in deeper water

y_* $y_1 - y_2$

r distance of the beach profile to a spiral center

Θ angle parameter in mathematical description of hooked bays

β spiral angle in mathematical description of hooked bays

a depth of hooked bays

b distance between headlands

MATHEMATICAL MODELING OF SHORELINE EVOLUTION

by

Bernard Le Mehaute and Mills Soldate

I. INTRODUCTION

This interim report presents a critical literature survey on the subject of mathematical modeling of shoreline evolution. Hopefully, this review will lead the way in establishing a flexible and practical numerical method suitable to predict shoreline evolution, resulting from the construction of navigation and shore protection structures in the Great Lakes.

To focus attention on the most pertinent literature, the subject under consideration is limited to long-term shoreline evolution as defined below.

Three time scales of shoreline evolution can be distinguished:

(a) Geological evolution taking place over centuries;

(b) long-term evolution from year-to-year or decade; and

(c) short-term or seasonal evolution and evolution taking place during a major storm.

Associated with these time scales are distances or ranges of influence over which changes occur. The geological time scale deals, for instance, with the entire area of the Great Lakes. The long-term evolution deals with a more limited stretch of shoreline and range of influence; e.g., between two headlands or between two harbor entrances. The short-term evolution deals with the intricacies of the surf zone circulation; e.g., summer profile-winter profile, bar, rhythmic beach patterns, etc.

For the problem under consideration, long-term evolution is of primary importance, the short-term evolution appearing as a superimposed perturbation on the general beach profile. Evolution of the coastline is characterized by low monotone variations or trends on which are superimposed short bursts of rapid development associated with storms.

The primary cause of long-term evolution is water waves or wave-generated currents. Three phenomena intervene in the action which waves have on shoreline evolution:

(a) Erosion of beach material by short period seas versus accretion by longer period swells;

(b) effect of (lake) level changes on erosion; and

(c) effect of breakwaters, groins, and other structures.

Even though mathematical modeling of shoreline evolution has inspired some research, it has received only limited attention from practicing engineers. The present methodology is based mainly on

(a) the local experience of engineers who have a deep knowledge of their sectors, understand littoral process, and have an inherent intuition of what should happen; and

(b) movable-bed scale models that require extensive field data for their calibration.

In the past, theorists have been dealing with idealized situations, rarely encountered in engineering practice. It seems that mathematical modelers have long been discouraged by the inherent complexity of the phenomena encountered in coastal morphology. The lack of well-accepted laws of sediment transport, offshore-onshore movement, and poor wave climate statistics have made the task of calibrating mathematical models very difficult.

Considering, on one hand, the importance of the subject of determining the effect of construction of long groins and navigation structures and on the other, the progress which has been made in determining wave climate and littoral drift, it now appears that a mathematical approach could be useful.

The complexity of beach phenomena could, to a large extent, be taken into account by means of numerical mathematical scheme, (instead of in closed-form solutions), dividing space and time intervals into small elements, in which the inherent complexity of the morphology could be taken into account.

Furthermore, better knowledge of the wave climate, a necessary input, will allow a better calibration of coastal constants such as found in the littoral drift formula.

This study emphasizes the relative importance of various reports and reviews the most important ones. Conclusions based on this review are presented, pointing out the deficiencies of the state-of-the-art. (Subsequent investigators should attempt to bridge the remaining gaps.)

The reports are presented individually, primarily in chronological order. Two milestone developments from this survey are reports by Pelnard-Considere (1956) and by Bakker (1968b). Others are extensions and refinements, experimental verifications, support papers, numerical procedures, and side issues, including the latest developments on "hooked beaches" or crenulate-shaped bays.

II. THE FIRST MODEL (PELNARD-CONSIDERE)

The idea of mathematically formulating shoreline evolution is attributed by Bakker (1968a) to Bossen, but no reference to Bossen is given. The first report which appears in the literature, on mathematical modeling of shoreline evolution, is by Pelnard-Considere (1956). His theoretical developments were substantiated by laboratory experiments made at Sogreah (Grenoble), France. The experimental results fit the theoretical results very well. It is surprising that such relatively simple theory has not been more frequently applied to prototype cases by the profession (at least as it would appear from the open literature), a fact which may be attributed to the lack of knowledge of wave climates.

Pelnard-Considere assumed that:

(a) The beach profile remains similar and determined by the equilibrium profile. Therefore, all contour lines are parallel. This assumption permits him to consider the problem to be solved for one contour line only.

(b) The wave direction is constant and makes a small angle with the shoreline ($<20°$).

(c) The longshore transport, Q, is linearly related to the tangent of the angle of incidence α ($Q \simeq f(\alpha)$, $f(\alpha) = \tan \alpha$).

(d) The beach has a fixed (ill-defined) depth, D (Fig. 1). D is a factor relating erosion retreat to volume removed from profile, which could be defined by the threshold velocity of sand under wave action. A practical method of determination of D is given in Section VIII.

Despite the crudeness of these approximations, the Pelnard-Considere model can be considered as a milestone in demonstrating the feasibility of mathematical modeling of long-term shoreline evolution. For this reason, it is judged useful to describe in some detail his theoretical development.

Consider an axis, ox, parallel to the main coastal direction and an axis, oy, perpendicular seawards (Fig. 2). The angle the deepwater wave makes with the axis, ox, is α_o. The angle of the wave with the shoreline α at any location is assumed to be small; therefore,

$$\alpha = \alpha_o - \tan^{-1} \frac{\partial y}{\partial x} \cong \alpha_o - \frac{\partial y}{\partial x} \quad \text{or} \quad \alpha - \alpha_o = - \frac{\partial y}{\partial x} \quad . \tag{1}$$

($y = f(x,t)$ gives the form of the shoreline as function of time t). The littoral drift Q is a function of angle incidence α and can be put into a Taylor series:

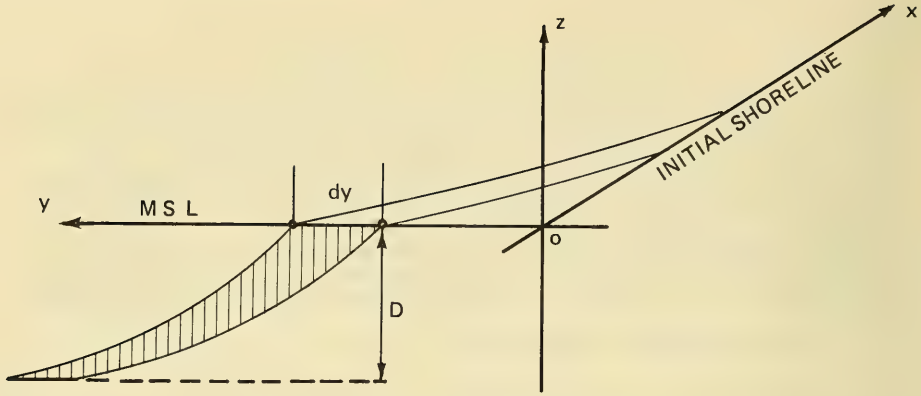

Figure 1. Beach depth definition.

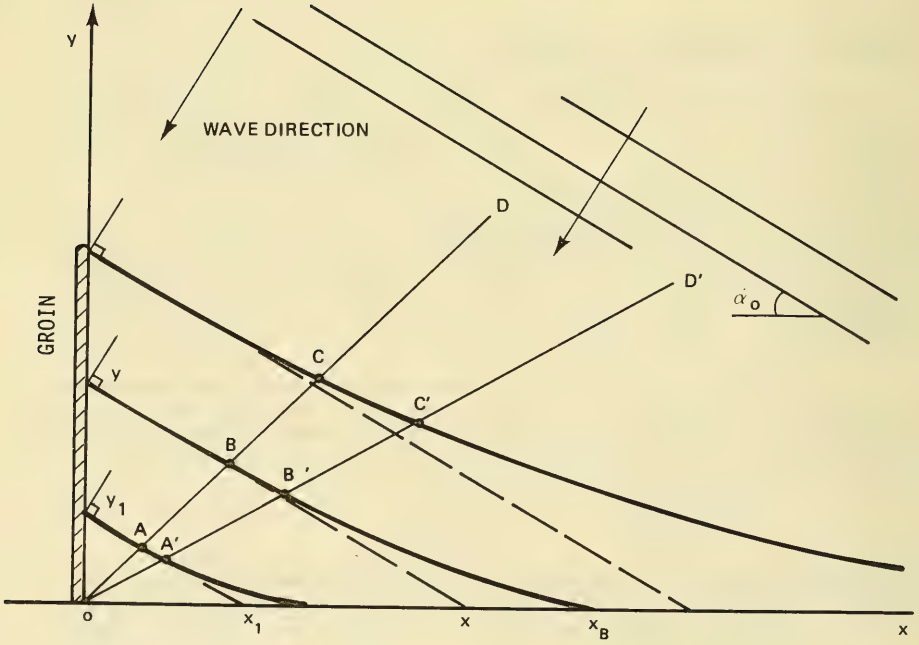

Figure 2. Successive beach profiles updrift of a long groin before bypassing (from LeMehaute and Brebner, 1961).

$$Q = Q_o + \left.\frac{\partial Q}{\partial \alpha}\right|_{\alpha = \alpha_o} (\alpha - \alpha_o) + \cdots \quad , \tag{2}$$

in which Q_o denotes the transport, Q, when the angle of the wave incidence is α_o. Substituting equation (1) into equation (2) yields:

$$Q \cong Q_o - \left[\left.\frac{\partial Q}{\partial \alpha}\right|_{\alpha = \alpha_o} \right] \frac{\partial y}{\partial x} \quad . \tag{3}$$

During the interval of time, dt, the shoreline recedes (or accretes) by a quantity dy. Therefore, the volume of sand which is removed (or deposited) over a length of beach, dx, is D dx dy. The quantity is equal to the difference of longshore transport during time, dt, between x and x + dx; i.e.,

$$Q \, dt \text{ and : and } (Q + \frac{\partial Q}{\partial x} dx) \, dt \; ;$$

i.e.,

$$\frac{\partial Q}{\partial x} \, dt \quad .$$

Therefore,

$$D \, dx \, dy = \frac{\partial Q}{\partial x} dx dt \quad , \quad \text{or} \quad \frac{\partial y}{\partial t} = \frac{1}{D} \frac{\partial Q}{\partial x} \quad . \tag{4}$$

Substituting the expression for Q, α being small, and defining

$$K = \frac{1}{D} \left.\frac{dQ}{d\alpha}\right|_{\alpha = \alpha_o} \tag{5}$$

yield:

$$K \frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t} \tag{6}$$

which is the well-known diffusion or heat-flow equation.

K is approximately constant at a given site. Bakker (1968a) found K equal to 0.4×10^6 cubic meters per meter depth per year, at an exposed site along the coast of the Netherlands. Equation (6) demonstrates that the rate of accretion or (erosion), $\frac{\partial y}{\partial t}$, is linearly related to the curvature of

the coast, the derivative of the longshore transport rate with respect to the angle of the wave incidence, $\left.\frac{dQ}{d\alpha}\right|_{\alpha = \alpha_o}$, and inversely proportional to the beach depth, D .

The above equation will be recognized as the well-known diffusion equation. A number of classical solutions of mathematical physics are applicable to the diffusion equation when boundary conditions are specified. Pelnard-Considere (1956) applied his theory to the case of a littoral barrier or long groin. This case is reviewed below:

The longshore transport rate along a straight, long beach is suddenly stopped by the construction of a long groin built perpendicular to the beach (see Fig. 2). The boundary conditions are:

(a) $y = o$ for all x when $t = o$ which characterizes an initial straight shoreline.

(b) At the groin, the longshore transport rate $Q = o$ which is realized when the waves approach the shore normally; i.e., when

$$\frac{\partial y}{\partial x} = -\tan \alpha_o \quad \text{at} \quad x = o \ ,$$

(c) $\frac{\partial y}{\partial x} = o$ at a large distance updrift ($x \to \infty$) , and $Q = Q_o$.

Q_o is the steady-state longshore transport along a straight beach for the given wave conditions. The solution for the given boundary conditions is:

$$y = \frac{\tan \alpha_o}{\sqrt{\pi}} \left[\sqrt{4Kt} \ \exp(-u^2) - x\sqrt{\pi} \ E(u) \right] , \qquad (7)$$

where $u = \frac{x}{(4Kt)^{1/2}}$, and $E(u)$ is the Fresnel integral,

$$E(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-u^2} \, du \ . \qquad (8)$$

Values of $E(u)$ or more frequently, $\phi(u) = 1 - E(u)$, can be found in tabulated form as given in Table 1:

Table 1. u versus ϕ (u).

u	0.1	0.2	0.3	0.4	0.5	0.7	0.9	1.2	2	∞
$\phi(u)$	0.112	0.223	0.328	0.428	0.520	0.667	0.796	0.910	0.995	1

Fig. 2 illustrates the shoreline evolution as defined by equation (8). It is interesting that these curves are homothetic with respect to the origin o ; i.e.,

$$\frac{oA}{oA'} = \frac{oB}{oB'} = \frac{oC}{oC'} \text{ , etc...}$$

The horizontal lengths grow with t , and in particular,

$$oy = \frac{\tan \alpha_o}{\sqrt{\pi}} \cdot 2 \sqrt{Kt} \text{ .}$$

A tangent to the shoreline at the groin intersects the initial shoreline defined by $y = o$ at a point a distance of $2 \sqrt{Kt/\pi}$ updrift from the groin.

The ratio of the area of sand accumulation, such as is in oyx_B , to the area of sand contained in the triangular fillet, oyx , is 1.56 and the distance $ox_B \cong 2.7$ ox . This ratio permits rapid assessment of the total amount of sand accumulated updrift from a single measurement of the angle α_o , and determination of D as shown in Section IV.

The end of the groin of length, $oy = \ell$, is reached when

$$t = t_1 = \frac{\ell^2 \pi}{4K \tan^2 \alpha_o} \text{ .} \qquad (10)$$

When $t \geq t_1$, the boundary conditions must be modified since the groin no longer traps all the sand but bypasses some of it.

If the same theory is applied to the beach downdrift of the groin and if assumed that the wave diffraction effects are negligible, the beach is eroded in a form symmetric with the updrift accretion.

When $t = t_1$, the end of the groin is reached by the shoreline and sand begins to be bypassed around the groin.

The boundary condition at the groin becomes $oy = \ell$ (constant) for $t > t_1$. The solution then becomes (Fig. 3):

$$y = \ell\, E\left[\frac{x}{\sqrt{4Kt}}\right]. \tag{11}$$

The curves representing the shoreline become homothetic with respect to the axis oy ; i.e.,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \text{, etc.}$$

The area between the shoreline and the ox axis $(oy_o x')$ is given by:

$$2\ell\sqrt{\frac{Kt}{\pi}} \quad .$$

The area of triangular fillet, $oy_o x_o$, is $\frac{\ell}{2}\sqrt{\pi Kt}$.

Hence,

$$\frac{oy_o x'}{oy_o x_o} = 2\ell\sqrt{\frac{Kt}{\pi}} \cdot \frac{1}{\frac{\ell}{2}\sqrt{\pi Kt}} = \frac{4}{\pi} = 1.27 \tag{12}$$

and

$$ox' = 2x_o \quad .$$

The shoreline as described by equation (7) at time $t = t_1$ is slightly different from the shoreline defined by equation (11) at $t = t_1'$ as shown in Fig. 4.

The volume of sand defined by both curves is equal when the time t_1 of equation (7) is replaced by the time t_1' in equation (11) in such a way that

$$\frac{t_1'}{t_1} = \frac{\pi^2}{16} \qquad \text{i.e.,} \quad t_1' = 0.62t_1 \quad . \tag{13}$$

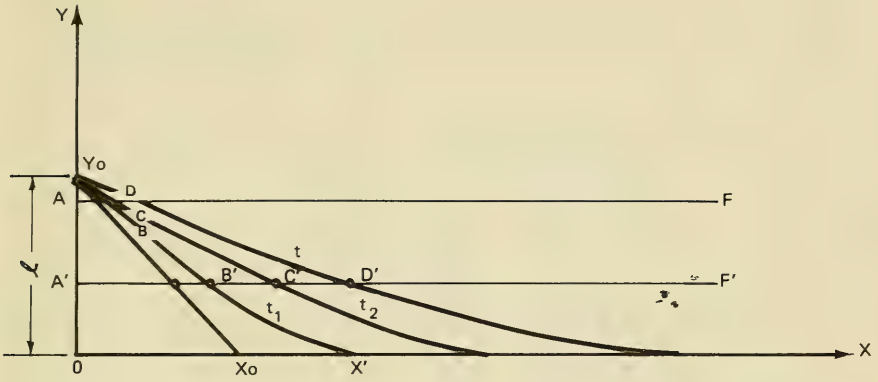

Figure 3. Successive beach profiles updrift of a groin after sand bypassing (from Le Mehaute and Brebner, 1961).

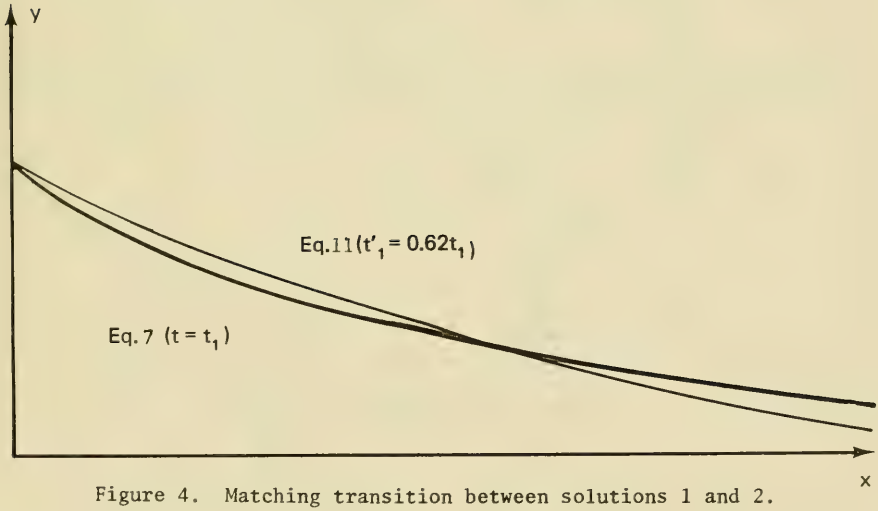

Figure 4. Matching transition between solutions 1 and 2.

Therefore, the shoreline evolves initially as represented by equation (7). Then, when $t = t_1$, the shoreline keeps evolving as given by equation (11), as if the time were $t - 0.38t_1$. Then, the sediment discharge, Q, bypassing the groin is equal to the incoming discharge Q_o minus the volume of sand which accumulates per unit of time.

$$Q(t) = Q_o - \frac{KD\ell}{\left[\pi K(t-0.38t_1)\right]^{1/2}} \quad ; \tag{14}$$

i.e.,

$$Q(t) = Q_o \left(1 - \frac{\ell}{\tan\alpha_o \left[\pi K(t-0.38t_1) \right]^{1/2}} \right) \tag{15}$$

or again

$$Q(t) = Q_o \left[1 - \frac{0.638}{\left[(t/t_1) - 0.38\right]^{1/2}} \right] \quad . \tag{16}$$

In dimensionless terms, the following values are obtained for equation (16) (see Fig. 5):

t/t_1	Q/Q_o
1	0.189
1.25	0.315
1.5	0.397
2	0.498
3	0.605
4	0.665
5	0.703

It takes a long time before the value of Q approaches initial discharge, Q_o, downdrift of the groin.

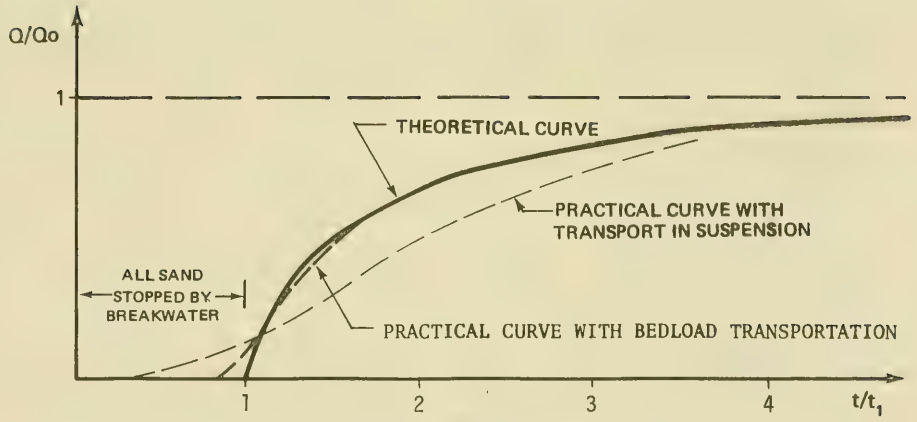

Figure 5. Sand bypassing long groin as a function of time (from Le Mehaute and Brebner, 1961).

The shoreline may be deduced at any time, t , by a homothetic transformation about the oy axis from the knowledge of the shoreline at a given time, t_2 , and also by applying the simple relationship (see Fig.3):

$$\frac{AD}{\left[t - 0.38t_1\right]^{1/2}} = \frac{AC}{\left[t_2 - 0.38t_1\right]^{1/2}} \qquad (17)$$

The theory of Pelnard-Considere has been verified in laboratory experiments with fairly good accuracy. The steady-state littoral drift, Q_o , was obtained experimentally from preliminary calibration over a straight shoreline. The results of these experiments are shown in Figs. 6 and 7. However, the shoreline predicted by theory is not expected to be valid downdrift of the groin because of the influence of wave diffraction around the groin tip. Some sand begins to bypass the groin by suspension before $t = t_1$ (see Fig. 5). Also, different boundary conditions apply to different contour lines since the deeper contour lines reach the end of the groin before the contour lines which are near the shoreline, which implies the one-dimensional theory is no longer entirely satisfactory.

Subsequently, Lepetit (1972) also conducted laboratory experiments which verify the results of a numerical scheme based on the theory of Pelnard-Considere. He used the law, $Q \propto \sin\alpha\sqrt{\cos\alpha}$. Lepetit's experiments were carried out with a very small angle between wave crest and shoreline.

1. Refinement and Extensions of the Pelnard-Considere Model.

After Pelnard-Considere's contribution, the mathematical formulation of shoreline evolution has proceeded at a slow pace. The first refinements came in improving the longshore transport rate (littoral drift) formula, in particular, modifying the expression relating sediment transport to incident wave angle.

Based on results from laboratory experiments performed by Sauvage and Vincent (1954), Larras (1957) introduced the function $f(\alpha) = \sin\frac{7\alpha}{4}$, also used by Le Mehaute and Brebner (1961). New theoretical forms of shoreline evolution are determined as solutions of the diffusion equation. Introduction of the relationship $f(\alpha) = \sin\frac{7\alpha}{4}$ instead of $\tan\alpha$, allows obtention of solutions valid for larger wave angles.

Of particular interest are the cases of shoreline undulations, since assuming linear superposition, any form of shoreline may be approximated by a Fourier series. The solution of the diffusion equation is then of the form:

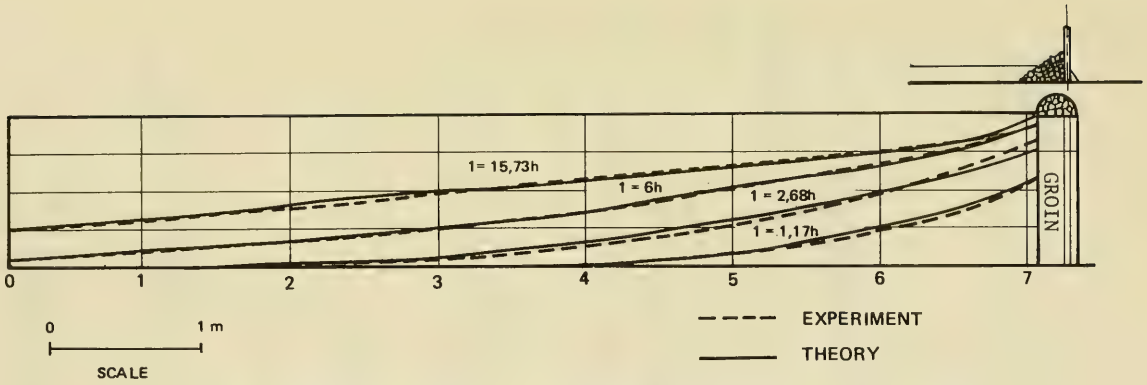

Figure 6. Comparison between experimental and theoretical shoreline evolution (from Pelnard-Considere, 1956).

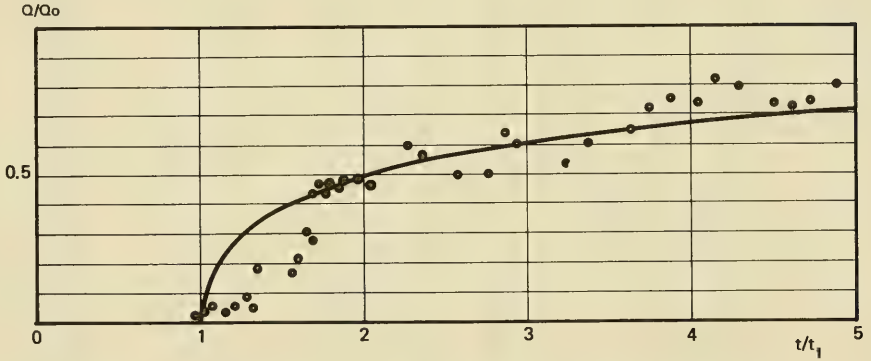

Figure 7. Comparison between experimental and theoretical sand bypassing discharge (from Pelnard-Considere, 1956).

$$y = Be^{-bt} \cos \sqrt{\frac{\lambda}{K}} \quad (x - x_o) \tag{18}$$

which indicates that shoreline undulations tend to decay exponentially and disappear with time. B defines the beach undulation amplitude at time, t = o , and λ is related to the wavelength, L , of this undulation through the relationship:

$$\lambda = \left(\frac{2\pi}{L}\right)^2 K \quad . \tag{19}$$

Shoreline evolution due to the sudden dumping of material at a given point may be represented by:

$$y = K \frac{e^{-x^2/4Kt}}{\sqrt{t}} \quad . \tag{20}$$

Equation (20) gives the spreading of the sand along the shoreline since the integration $\int_{-\infty}^{\infty} y dx$, which expresses the conservation of sediment in the system, is a constant (see Fig. 8). This solution was also mentioned by Pelnard-Considere.

It is interesting that much later, Noda (personal communication, 1974) investigated the same problem by taking an initial condition for sand dumping.

$$y = f(x,o) = \begin{cases} Y = \text{constant when } |x| < X \\ o \qquad\qquad \text{when } |x| > X \end{cases}$$

as shown on Fig. 9. Using the functional relationship now commonly accepted, $f(\alpha) = \sin 2\alpha$, Noda found that the solution to the diffusion equation to be:

$$y = \frac{Y}{2} \left\{ \text{erf} \left[\frac{(X - x)}{2\sqrt{Kt}} \right] - \text{erf} \left[\frac{(-X + x)}{2\sqrt{Kt}} \right] \right\} \quad . \tag{21}$$

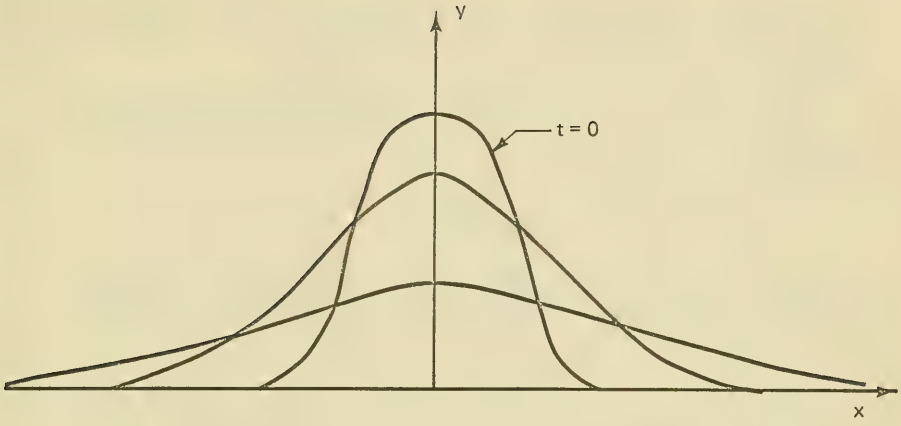

Figure 8. Spreading of sand along a shoreline due to instantaneous dumping at a point.

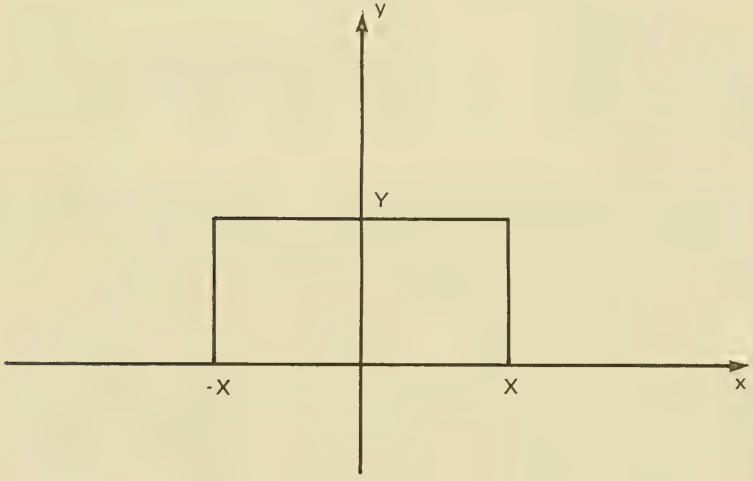

Figure 9. Sand dumping along a finite stretch of beach (initial condition).

Even though the initial condition is different from the previous one, the solutions tend to be similar as time increases and are, therefore, both applicable to the problem of shoreline sand dumping.

Also of interest is the solution, proposed by Larras (1957), of a beach equilibrium shape between two headlands or groins described by the equation:

$$\frac{\partial Q}{\partial s} = 0 ,$$

where s is the distance along the shoreline. This indicates no sand transport along shoreline configuration and, therefore, yields an equilibrium to obtain:

$$ds = L \cos \frac{7\alpha}{4} \, d\alpha \quad \text{(where } L \text{ is a proportionality constant)},$$

which gives

$$x = R \left[\sin \frac{11\alpha}{4} + \frac{11}{3} \sin \frac{3\alpha}{4} \right] \tag{22}$$

$$y = -R \left[\cos \frac{11\alpha}{4} + \frac{11}{3} \cos \frac{3\alpha}{4} \right] .$$

Equation (22) defines a hypocycloidal form as might be found between two headlands (see Fig. 10). R is a parameter which is related to the relative curvature of the shoreline. When $R \to \infty$, a straight shoreline solution is obtained.

Another family of solutions was given by Grijm (1960, 1964). In these two publications, Grijm used the most commonly accepted expression for dependence of longshore transport on angle, $f(\alpha) = \sin 2\alpha$, and applied the theory to cases where the angle of incidence, α, is not necessarily small. Subsequently, he established the kind of shoreline which can exist mathematically under steady-state conditions.

Even though the theoretical approach obeys the same physical assumption as the previous theory (except for the allowable range for the angle of incidence), his mathematical formulation is not as simple. The shoreline is defined with respect to a polar coordinate axis. The continuity equation is solved in parametric form, which is integrated either by computer or by graphical methods. Details of Grijm's computations are not available.

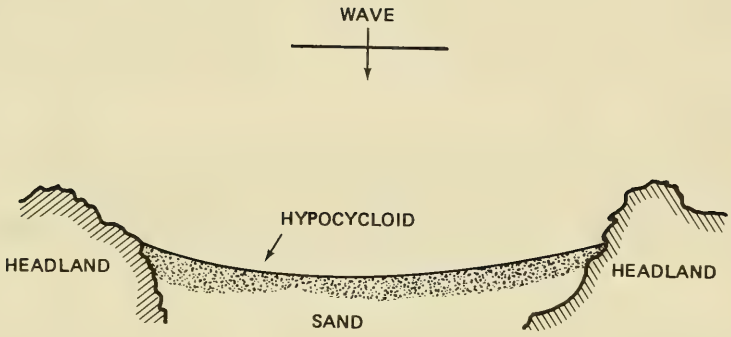

Figure 10. Equilibrium profile between two headlands.

The main interest of the report lies in the results. When the longshore transport rate reaches a maximum value ($\alpha = 45°$), the shoreline tends to form a cusp; i.e., a cape as shown in Fig. 11.

Also of interest is Grijm's (1964) mathematical formulation for different forms of river deltas for which he finds two possible solutions, one with an angle of wave incidence everywhere less than 45°, and another with the angle of incidence greater than 45°. The shoreline curvature also depends upon the angle α as shown on Fig. 11. The problem remains indefinite since it is unknown which solution is valid.

The formulation of Grijm does not lend conveniently to numerical adaptation.

Bakker and Edelman (1964) also studied the form of river deltas, but instead of using $f(\alpha) = \sin 2\alpha$, as Grijm, they used the linear approximation as given by Pelnard-Considere; i.e., $f(\alpha) \cong k_1 \tan\alpha$ for $0 < \tan\alpha < 1.23$. They also investigated the case of large angle of approach using the function:

$$f(\alpha) = \frac{k_2}{\tan\alpha} \quad \text{for} \quad 1.23 < \tan\alpha < \infty \quad .$$

Bakker and Edelman's (1964) solutions are similar to that of Grijm; however, they also found a periodic solution as Larras (1957) did:

$$y = \exp\left\{ \left[\frac{-dQ}{d\alpha} \;\middle|\; \alpha = \alpha_o \right] \frac{1}{D} K^2 t \right\} \cos kx \quad . \qquad (23)$$

Equation (23) represents a sinusoidal shoreline for which the amplitude of the undulations decreases with time if $\frac{dQ}{d\alpha}$ is positive (i.e., for small angles of wave incidence), but increases when $\frac{dQ}{d\alpha}$ is negative (i.e., for large angle of wave incidence). The shoreline is thus unstable and the amplitude of the undulations increases. It can be deduced that Grijm's solution for large angles of incidence is not naturally found, since they are unstable and will be destroyed as small perturbations trigger large deviations.

Bakker (1968a) implies that Grijm did not discover this instability because he confined himself to solutions growing linearily with t in all directions, while the exponential solution in t also exists. Komar (1973) also applies a numerical scheme based on the Pelnard-Considere approximation to the problem of delta growth. He found shoreline shapes identical to Grijm in the case of a small angle of approach.

From these investigations, it is remembered that the Pelnard-Considere approach is very powerful to predict shoreline evolution under small angle of incidence. But under large angle of incidence, instability of the shoreline makes it very difficult. Furthermore, the

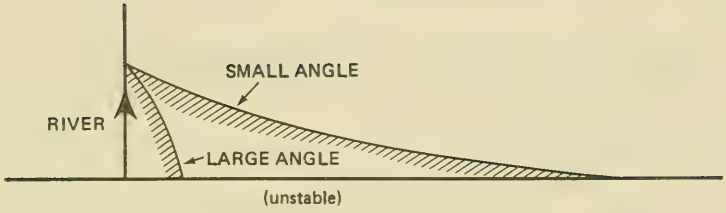

Figure 11. Two theoretical forms of shoreline equilibrium of river deltas.

phenomenology of interaction between wave and shoreline is not accurately defined mathematically.

2. Example of Shoreline Evolution.

Because of its importance, an example application of the theory of shoreline evolution is presented. However, the example is slightly modified to account for the generally accepted longshore transport rate formula:

$$Q = \frac{k}{16} \rho g \, H_b^2 \, C_g \sin 2\alpha_b \quad , \tag{24}$$

where

Q = longshore transport rate cubic feet per second

α_b = wave breaking angle

H_b = breaking wave height

C_g = wave group velocity at breaking

k = a constant $\simeq 6.42 \times 10^{-3}$

ρg = specific weight of seawater.

For the case of a groin perpendicular to shore, consider the average beach conditions:

H_b = 5 feet

d_b = 6.4 feet

α_o = 5°

D = 20 feet

$C_g = \sqrt{gd_b}$ = 14.4 feet per second

Thus,

$$K = \frac{2}{D} \frac{k}{16} \rho g \, H_b^2 \, C_g \cos 2\alpha_b$$

$$= \frac{(6.42 \times 10^{-3}) \quad (64) \quad 5^2 \quad (14.4)}{8 \times 20} = 0.92 \quad .$$

Substituting into equation (10), yields:

$$t_1 = \frac{\ell^2 \pi}{4K \tan^2 \alpha_o} = 1.3 \times 10^{-3} \ \ell^2 \text{ days.}$$

In tabular form for various groin lengths,

ℓ ft	50	100	200	500
t_1 days	3	13	52	325

Check:

For ℓ = 50 feet Area $Ox_b y$ = 1.56 x Area oxy

$$= 1.56 \frac{\ell^2}{2 \tan\alpha_o} = \frac{0.78 \ \ell^2}{\tan\alpha_o} = 22{,}400 \text{ square feet}$$

Volume $\cong$ (Area $ox_b y$) (D) = 4.5×10^5 cubic feet

$$Q = \frac{KD}{2} \tan 2\alpha_b = 1.6 \text{ cubic feet per second}$$

$$t_1 = \frac{4.5 \times 10^5}{Q} = 2.8 \times 10^5 \text{ seconds} \cong 3 \text{ days.}$$

III. THE TWO-LINE THEORY OF BAKKER

One limitation of the solutions of Pelnard-Considere is the assumption of parallel depth contours. Bakker (1968a) realized that the one-line theory of Pelnard-Considere and its subsequent development may, at times, lead to some inaccuracy, since beach slope variations along the shore were not considered. Beach slope variations with respect to time (summer-winter profiles) are not important in the long-term shoreline evolution. Nevertheless, if an adequate onshore-offshore profile response model was available, a suitable mathematical representation of it could be developed (Dean, 1973; Swart, 1974).

Near coastal structures, the deviations of the model from prototype conditions can be considerable. Pelnard-Considere finds that the accretion and erosion patterns are symmetrical with respect to the groin as shown on Fig. 12. However, in reality, the updrift profile becomes steeper than the equilibrium profile and the sand moves seaward. The downdrift profile is flatter than the equilibrium profile and the sand

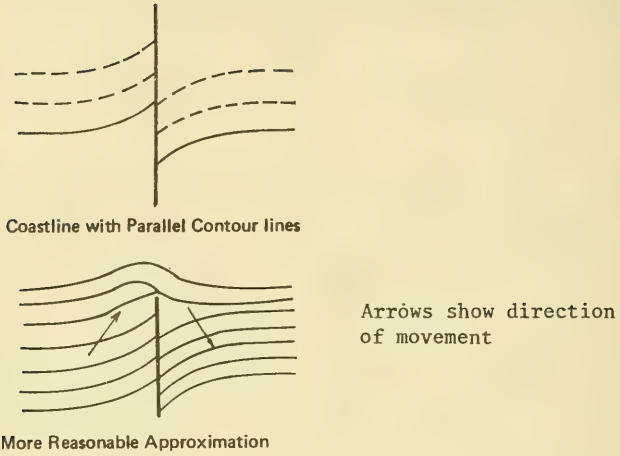

Figure 12. Differences on shoreline configuration due to onshore-offshore transport near a groin (from Bakker, 1968b).

is pushed shoreward by the waves. To reproduce the onshore-offshore movement in a mathematical model, it is necessary to schematize the coast by two or more contour lines instead of one.

Bakker's (1968b) two-contour-line model is not easily applied to practical engineering problems encountered by designers, due to lack of knowledge about onshore-offshore transport. However, his contribution toward establishing a realistic mathematical model of shoreline evolution is of sufficient importance to deserve detailed review.

Bakker (1968b) assumes that the profile is divided into two parts (Fig. 13). The upper parts extending to a depth, D_1 , are affected by the groin, the part below D_1 extends offshore to a depth of $D_1 + D_2$ which is the assumed practical seaward limit of material movement.

The "equilibrium distance", w , is defined by a distance $(y_2' - y_1)$ corresponding to an equilibrium profile under normal conditions; i.e., far away from the groins.

The onshore-offshore transport is defined by:

$$Q_y = q_y \quad (y_1 - y_2' + w) \quad , \tag{25}$$

where q_y is a proportionality constant (dimension LT^{-1}). When $(y_1 - y_2' + w)$ is positive, the transport is seaward; when negative, it is shoreward. q_y has been found by Bakker for a part of the Dutch coast equal to 1 to 10 meters per year for a depth $D_1 = 3$ meters. Letting $y_2 = y_2' - w$, then, $Q_y = q_y (y_1 - y_2)$.

Now, following Pelnard-Considere; i.e., developing the expression for the longshore transport rate Q in a Taylor series in terms of α ,

$$Q = Q_o + \frac{\partial Q}{\partial \alpha} \quad (\alpha - \alpha_o) \quad + \ldots \tag{26}$$

which gives in linear approximations:

$$Q = Q_o - \left[\frac{dQ}{d\alpha} \Bigg|_{\alpha = \alpha_o} \quad \frac{\partial y}{\partial x} \right] \quad . \tag{27}$$

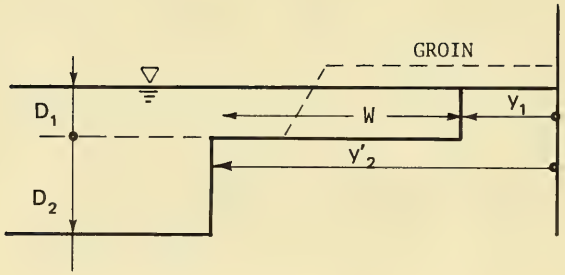

Figure 13. Notation for the two-line theory.

Defining

$$q = \left[\frac{dQ}{d\alpha} \Bigg| \alpha = \alpha_o \right] \quad , \tag{28}$$

then,

$$Q = Q_o - q \frac{\partial y}{\partial x}$$

This equation is now applied to both lines, $y_1(x)$ and $y_2(x)$:

$$Q_1 = Q_{o1} - q_1 \frac{\partial y_1}{\partial x} \tag{29}$$

$$Q_2 = Q_{o2} - q_2 \frac{\partial y_2}{\partial x} \quad . \tag{30}$$

The equation of continuity,

$$\frac{\partial Q}{\partial x} + D \frac{\partial y}{\partial t} = 0 \quad , \tag{31}$$

is modified by the term Q_y due to onshore-offshore transport so that

$$- \frac{\partial Q_1}{\partial x} - Q_y = D_1 \frac{\partial y_1}{\partial t} \tag{32}$$

$$- \frac{\partial Q_2}{\partial x} + Q_y = D_2 \frac{\partial y_2}{\partial t} \quad . \tag{33}$$

Substituting equations (1), (2), and (3) for Q_1, Q_2, Q_y gives:

$$q_1 \frac{\partial^2 y_1}{\partial x^2} - q_y (y_1 - y_2) = D_1 \frac{\partial y_1}{\partial t} \tag{34}$$

$$q_2 \frac{\partial^2 y_1}{\partial x^2} - q_y (y_2 - y_1) = D_2 \frac{\partial y_2}{\partial t} \quad . \tag{35}$$

Adding both equations yields:

$$\frac{q}{D} \frac{\partial^2 y}{\partial x^2} + \frac{D_1 D_2}{D^2} \left(\frac{q_1}{D_1} - \frac{q_2}{D_2} \right) \frac{\partial^2 (y_1 - y_2)}{\partial x^2} = \frac{\partial y}{\partial t} \tag{36}$$

in which

$$\frac{q}{D} = \frac{q_1 + q_2}{D_1 + D_2} \quad \text{and} \quad y = \frac{1}{D} (D_1 y_1 + D_2 y_2) \quad . \tag{37}$$

For simplicity, Bakker (1968b) assumes $\frac{q_1}{D_1} = \frac{q_2}{D_2}$, which implies that derivatives of the littoral drift transport with respect to α: $\left. \frac{dQ}{d\alpha} \right|_{\alpha = \alpha_o}$ are proportional to depth D. Then, dividing equation (6) by D_1 and D_2 respectively, and subtracting, yield :

$$\frac{q}{D} \frac{\partial^2 (y_*)}{\partial x^2} - \frac{q_y D}{D_1 D_2} (y_*) = \frac{\partial (y_*)}{\partial t} \tag{38}$$

where $y_* = y_1 - y_2$, which is the equation for the offshore-onshore transport $q_y y_*$. It is interesting that the offshore-onshore transport is independent of the longshore transport.

Using the auxiliary variable,

$$y_e = y_* \exp \left(\frac{q_y D}{D_1 D_2} t \right) \quad , \tag{39}$$

the diffusion equation is still obtained:

$$\frac{q}{D}\frac{\partial^2 y_e}{\partial x^2} = \frac{\partial y_e}{\partial t} \quad . \tag{40}$$

Bakker has applied his theory to a number of idealized cases, including the behavior of a sand beach near a groin, assuming

$$\left\{ \begin{array}{l} D_1 = D_2 \\ q_1 = q_2 \end{array} \right\} \quad . \tag{41}$$

The boundary conditions are:

a. Initial condition (t = o): $y_1 = y_2 = o$ for $o < x \leq \infty$ and $t = o$.

b. Then, when $t > o$:

(1) $y_1 = y_2 = o$ for $x \to \infty$ and $o < t \leq \infty$ (which implies an equilibrium profile)

(2) $y_2 = o$ for $x = o$

(3) $\frac{\partial y_1}{\partial x} = \tan\alpha \frac{Q_{o1}}{q_1}$ for $x = o$.

The results are expressed in terms of lengthy power series, and are represented graphically in Fig. 14.

The case of equilibrium beach profiles between groins was also investigated by Bakker (1970).

Despite the complex refinement of the two-line theory, as initially developed by Bakker, a number of phenomena that have significant influence on the beach profile are still neglected. Among these are:

a. The influence of rip current near the groins is twofold: rip currents transport material from beach to the offshore and cause wave refraction.

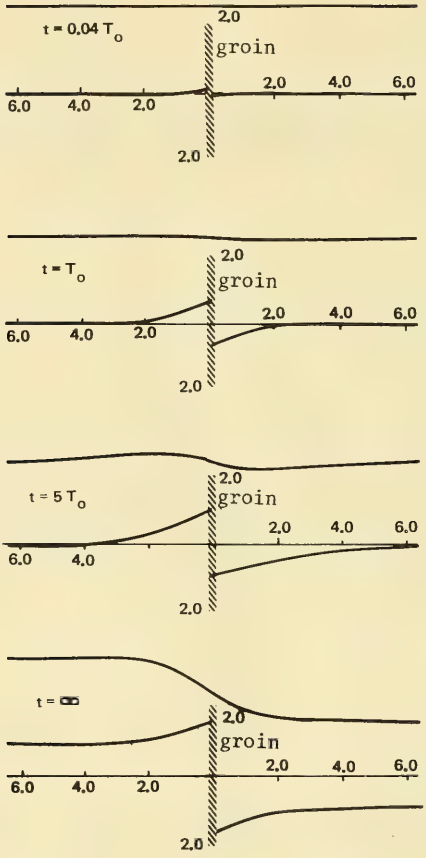

Figure 14. Evolution of shoreline and offshore beach limit near a groin (from Bakker, 1968b).

b. The influence of diffraction on the leeward side of groins which has an effect in the immediate vicinity of the shoreline.

c. The effect of changing wave direction caused by refraction changes the magnitude of longshore transport rate and the boundary conditions.

d. The nonlinearity in the transport equation is of minor importance for small angles of incidence (for $\alpha > 45°$, the coastline becomes unstable as previously mentioned).

The two-line theory has been verified experimentally (Hulsbergen, Van Bochove, and Bakker, 1976), and shows a trend identical to the experimental results. There are some differences at a small scale due to secondary currents, breaking wave type, changes of wave height due to small changes in morphology, etc. These, however, are short-term rather than long-term evolution phenomena.

IV. THE EFFECT OF WAVE DIFFRACTION

The effect of wave diffraction was subsequently taken into account by Bakker (1970). Initially, this was done for the one-line theory of Pelnard-Considere and later for Bakker's two-line theory.

Pelnard-Considere's equations,

$$Q = Q_o(x) - q(x) \frac{\partial y}{\partial x} \quad , \quad q(x) = \frac{\partial Q_o}{\partial \alpha} \Bigg|_{\alpha = \alpha_o} \tag{42}$$

and

$$\frac{\partial y}{\partial t} = - \frac{q(x)}{D} \frac{\partial Q(x)}{\partial x} \tag{43}$$

still apply. Q_o and q are now functions of x, since both the incident wave height and angle of approach vary along the shore with x, because of wave diffraction.

Inserting the expression for Q in the continuity equation, yields:

$$\frac{\partial y}{\partial t} = \frac{1}{D} \left(q \frac{\partial^2 y}{\partial x^2} + \frac{\partial q}{\partial x} \frac{\partial y}{\partial x} \right) - \frac{1}{D} \frac{dQ_o}{dx} \quad . \tag{44}$$

It is assumed that the longshore transport rate, Q_o, is proportional to the angle of wave incidence, $(\alpha_x - \frac{\partial y}{\partial x})$, and the square of the relative wave height. The variation of wave height with x is given by the diffraction theory of Putnam and Arthur (1948). The modification of wave diffraction by wave refraction is neglected.

A similar approach has been proposed by Price, Tomlinson, and Willis (1972), who assume that $Q = \frac{0.35}{\gamma_s} E \sin 2\alpha$, where E is the transmitted energy which is also a function of x as is α (and γ_s is the submerged density of the beach material). Price, Tomlinson, and Willis then obtain the one-line theory equation:

$$\frac{0.35}{\gamma_s} \left[\sin 2\alpha \frac{dE}{dx} + 2E \cos 2\alpha \frac{d\alpha}{dx} \right] + D \frac{\partial y}{\partial t} = o \tag{45}$$

which is solved numerically with

$$\alpha = \alpha_x - \tan^{-1} \frac{\partial y}{\partial x} \tag{46}$$

Laboratory experiments were performed with crushed coal by Price, Tomlinson, and Willis (1972). The theory giving the effect of wave diffraction was verified by the experiments at the beginning of the test. After a 3-hour test which may correspond to a prototype storm duration, it is stated that the wave refraction pattern invalidates the input wave data and a complex boundary condition developed at the updrift end of the wave basin.

Bakker's (1970) consideration of wave diffraction has been included in his two-line theory where,

$$Q_1 = Q_{o1} - q_1 \frac{\partial y_1}{\partial x} \tag{47}$$

$$Q_2 = Q_{o2} - q_2 \frac{\partial y_2}{\partial x} \tag{48}$$

Neither the deepwater line, defined by $y_2(x,t)$, nor q_2 and Q_{o2}, is affected by diffraction. Fig. 15 presents typical results obtained from this theory for the case of beach evolution near a groin and between two groins.

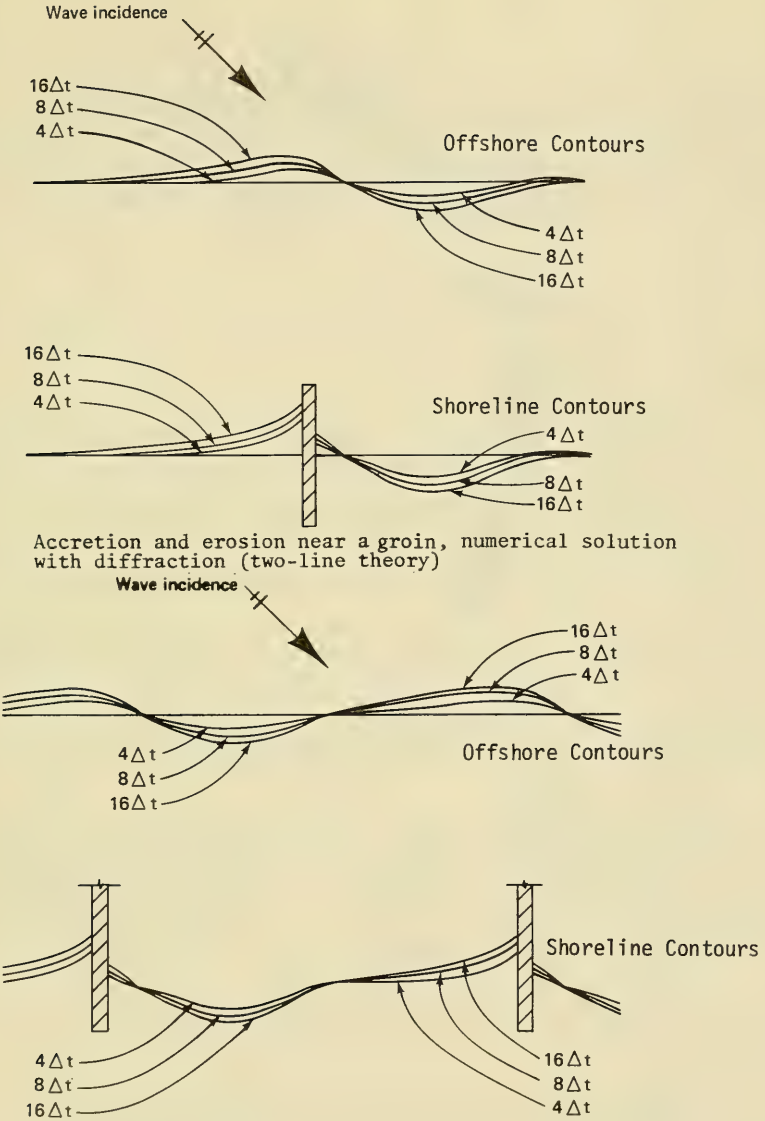

Figure 15. Effect of wave diffraction (from Bakker, 1970).

V. SPIRAL BEACHES

Hooklike beaches (Fig. 16) are common along exposed coasts and are formed by the long-term combined effects of refraction and diffraction around headlands. Yasso (1965) discovered that the planimetric shape of many of these beaches could be fitted very closely by a segment of logarithmic spiral; the distance, r, from the beach to the center of the spiral increasing with the angle θ according to

$$r = r_o \exp \left[\theta \cot \beta \right] \tag{49}$$

in which β is the spiral angle.

Bremmer (1970) has shown the logarithmic spiral to give an excellent fit for the profile of a recessed beach between two headlands.

The evolution of spiral beaches belongs to the geographical time-scale domain (Sylvester and Ho, 1972). However, similar evolution has also been observed over smaller time scales in consonance with the definition of long-term shoreline evolution adopted in this study.

So far, only empirical rather than theoretical mathematical representations of spiral beaches are available. The empirical approach has been fruitful in providing the spiral coefficients β as function of wave angle, α, with the headland alinement (Fig. 16) (Sylvester and Ho, 1972). The "indentation ratio" (depth of the bay to width of opening) also depends upon α and, in most cases, varies between 0.3 and 0.5 (Fig. 17).

There have been many attempts to explain this peculiar beach formation (Leblond, 1972; Rea and Komar, 1975). Leblond assumed that the rate of sediment transport is proportional to the longshore currents as given by the theory of Longuet-Higgins (1975). He also assumed that the beach profile is not modified by erosion or accretion so that the continuity equation from the one-line theory can be used in a two-dimensional coordinate system.

Thus, the variation in longshore current intensity with wave angle will yield the rate of erosion or accretion.

Difficulties arise in expressing this variation of longshore current in areas subjected to wave diffraction. Leblond (1972) points out that classical wave diffraction theories are too complicated to be used in his theoretical scheme. Another difficulty arises from the fact that the barrier (headland) is not thin as it is assumed in the theory of diffraction of Putnam and Arthur (1948). To account for this effect, Leblond introduces an empirical correction coefficient to the theory of Putnam and Arthur over a two-dimensional network. The results of

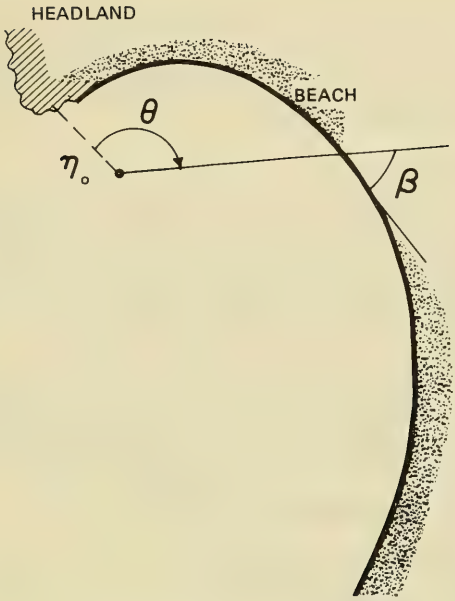

Figure 16. Hooked beaches.

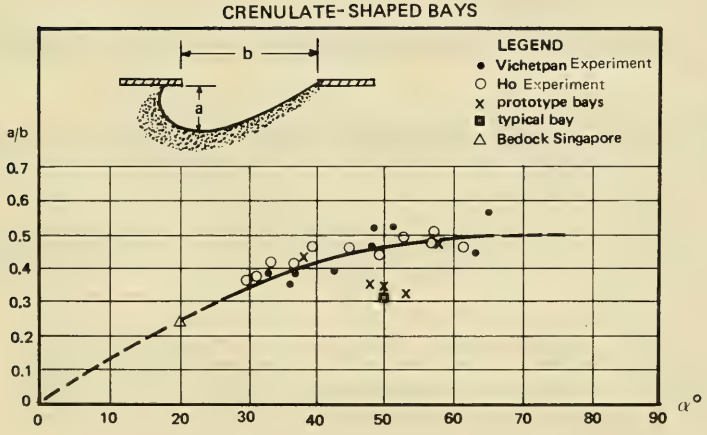

Figure 17. Indentation ratio for a range of wave obliquity (from Sylvester and Ho, 1972).

such a complex scheme, which is plagued with numerical instabilities, are shown in Fig. 18. Even though the results show how oblique waves initiate an erosion pattern that might eventually lead to the formation of hooklike beaches, they do not show that the beaches represent a good fit to segments of a logarithmic spiral.

Rea and Komar (1975) developed an approach to overcome the numerical instability encountered by Leblond. They combined two orthogonal, one-dimensional arrays as shown on Fig. 19. In this way, deformation of the beach can proceed in two directions without the necessity of a two-dimensional array. The wave configuration in the shadow zone was described by various simple empirical functions which resulted in beach configurations fairly approximated by a logarithmic spiral.

The main interest in the work of Rea and Komar (1975) is that they show the lack of sensitivity of the shoreline evolution in the shadow zone to the actual pattern of incident waves used. Also, the sensitivity of the beach shape to the energy distribution seems to be small.

VI. PROTOTYPE APPLICATIONS

The application of mathematical models of shoreline evolution to prototype conditions is not very well documented in the literature. It is certain that, at least in its simplified form such as given by Pelnard-Considere, the method has been used by practicing engineers and designers. It has been reported in unpublished reports but very little has appeared in the open literature.

Weggel (1976) has formulated a numerical approach to coastal processes which is particularly adapted to prototype situations. In particular, it includes:

a. A method for determining the water depth beyond which the onshore-offshore sediment transport is negligible. This information is particularly useful in determining the quantity D used in Pelnard-Considere's theory and others. It is also useful in determining the effect of a change of sea level. Beach profile data are plotted on semilog paper and the base elevation of the most seaward point varied until an approximate straight line is obtained (see Fig. 20). He found $D = 70$ feet at Pt. Mugu, California.

b. The effect of a change in sea level, a situation pertinent to the Great Lakes, is also taken into account in a way proposed by Bruun (1962). Using the principle of similarity of shoreline profile, the shoreline recession Δy is related to the change of water level a by the relationship (Fig. 21):

$$\Delta y = \frac{ab}{(\ell + d)} \tag{50}$$

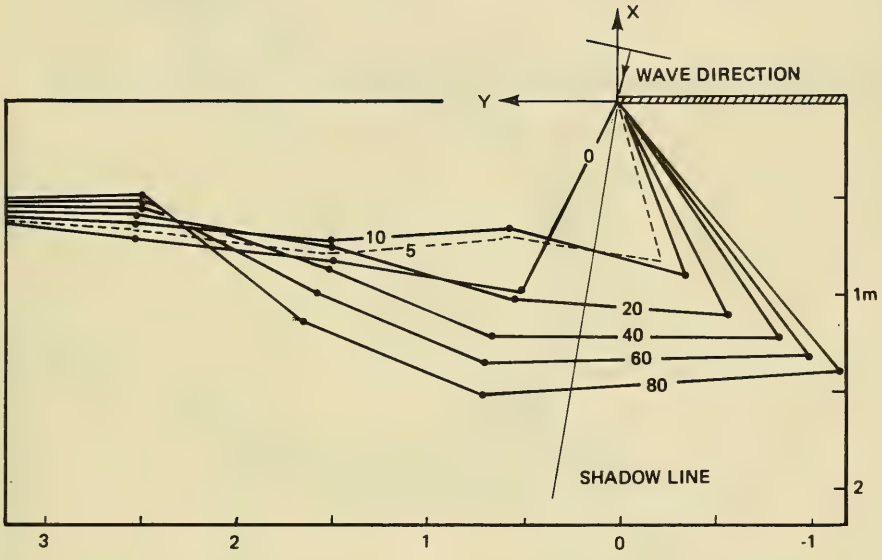

Figure 18. Orthogonal arrays for numerical scheme of hooked bay (from Leblond, 1972).

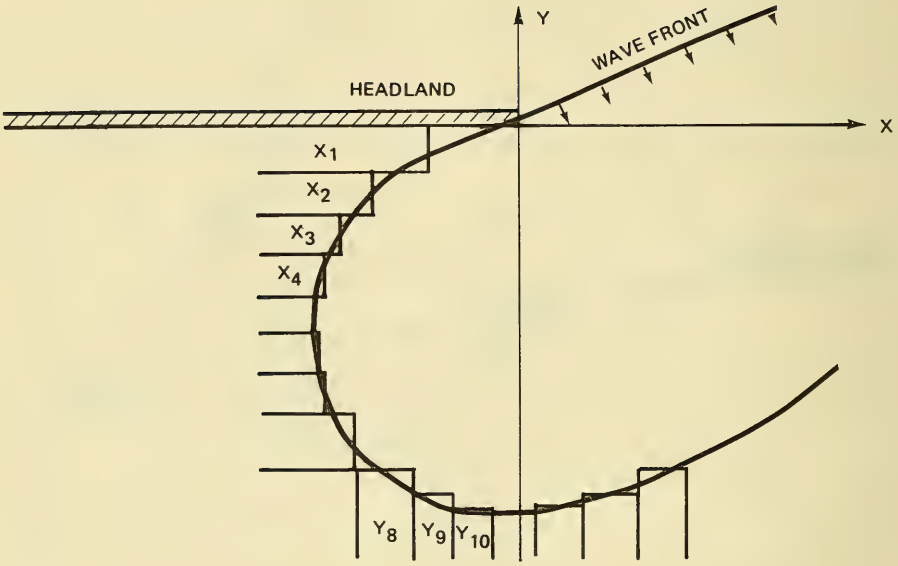

Figure 19. Orthogonal arrays for numerical scheme of hooked bay (from Rea and Komar, 1975).

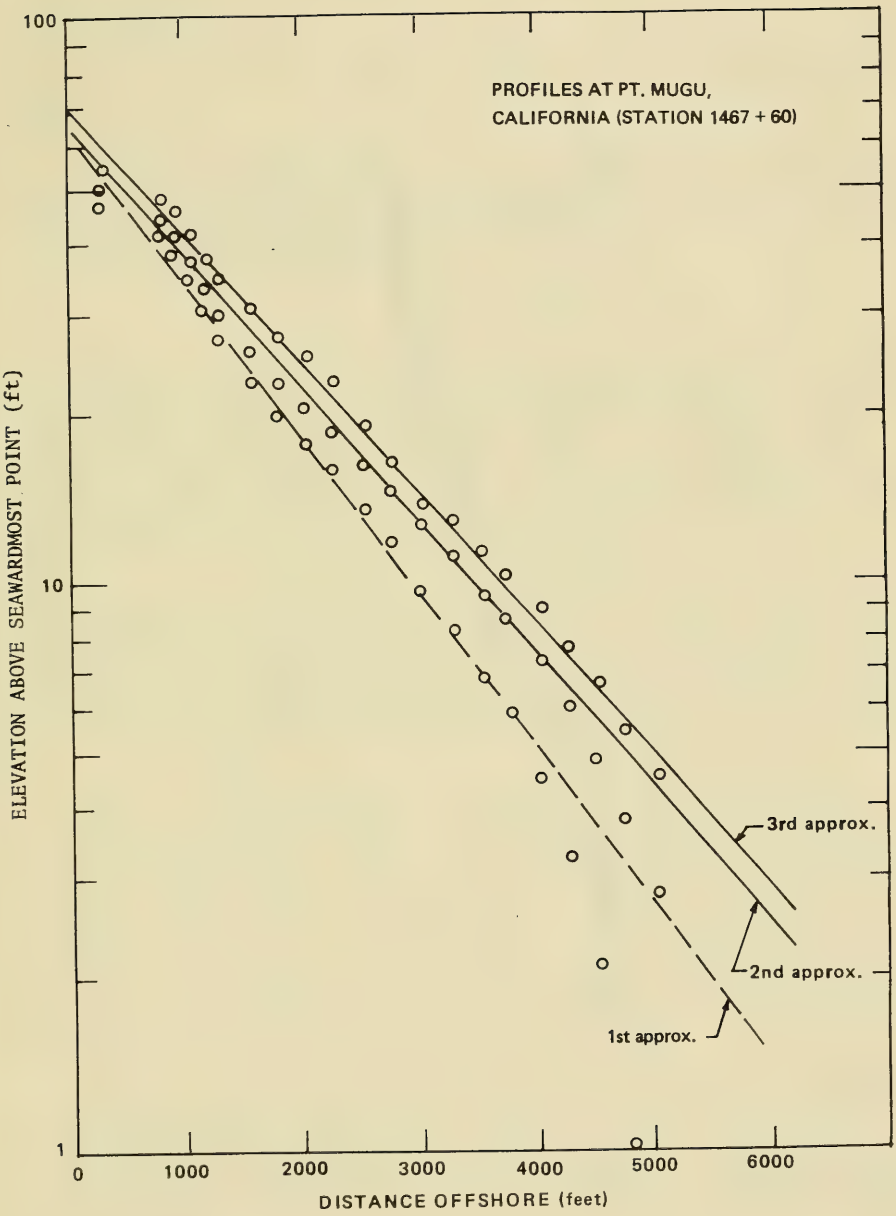

Figure 20. Semilogarithmic profiles (from Weggel, 1976).

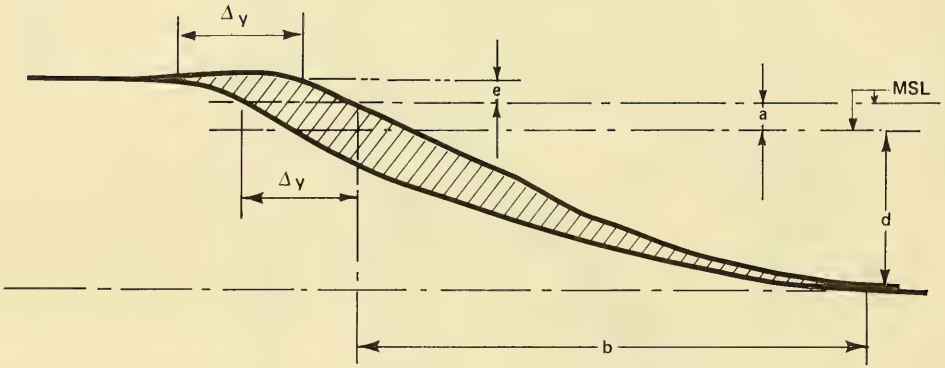

Figure 21. Relationship between shoreline retreat and change in mean water level.

c. A numerical scheme in which the effect of wave diffraction could be included.

d. A statistical characterization of wave climate and longshore energy flux.

Examples of recent prototype analysis and prediction of shoreline evolution by mathematical modeling are Apalachicola Bay by Miller (1975) and the Oregon coastline by Komar, Lizarraga-Arciniega, and Terich (1976). Both studies are based on numerical schemes related to the Pelnard-Considere (one-line) formulation.

VII. CONCLUSIONS

There are two methods of approach to the problems related to littoral processes. The first one, typified by the previously discussed reports, consists of analyzing global effects. The method essentially based on establishing "coastal constants" for a model by correlation between long-term evolution and wave statistics and subsequently, to use the model for predicting future effects. It appears that this method is the most promising for engineering purposes and could be termed the macroscopic view. The main results are summarized in Table 2.

The second approach, the microscopic view of the problem, consists in analyzing sediment transport, step-by-step, on a rational Newtonian approach, starting with wave motion, threshold velocity for sand transport, equilibrium profiles of beaches, etc., until the individual components can be combined into an overall model to predict shoreline evolution. The second method or scientific approach has not progressed to the point where it can be applied to engineering problems in the foreseeable future.

However, much progress has been made in the last 5 years toward understanding the hydrodynamics of the surf zone through application of the "radiation-stress" concept. In theory, establishing a reliable mathematical model of surf zone circulation should permit a determination of the resulting sediment transport. Practically, however, interaction between a movable bed and the surf zone circulation, and the inherent instability of longshore currents limit this approach to the realm of research. Among the problems that make this approach difficult are the refraction and diffraction of water waves, uncertainty in predicting rip current spacing, and the effect of free turbulence generated by breaking waves on the rate of sediment suspension.

Finally, the complexity of mathematical formulation, based on the radiation-stress concept, makes it difficult to use as a predictive tool when dealing with forcing functions expressed by statistical multidirectional sea spectra. This method is promising in explaining local effects (e.g., near groins), rhythmic topography, beach cusps, and short-term evolution due to unidirectional sea states. All these effects are

Table 2. Summary of mathematical models for shoreline evolution.

Date	Author	Sediment transport alongshore	Validity	Sediment transport onshore-offshore	Theoretical developments	Experimental verification	Application to ideal cases
1956	Pelnard-Considere	$\tan\alpha \approx \alpha = \alpha_o - \frac{\partial y}{\partial x}$	Very small angle	No	Diffusion equation closed-form solution	Laboratory with pumice	Groins
1957	Larras	$\sin\frac{7\alpha}{4} \approx \alpha = \alpha_o - \frac{\partial y}{\partial x}$	Small angle ($<25^o$)	No	Diffusion equation, closed-form solution	No	Groins-sudden dump sinusoidol undulation, equilibrium shape between groins
1960	Grijm	$\sin 2\alpha$	Sin all angles. In case of large angle an inconsistency in the assumption $\frac{\partial y}{\partial x}$ small	No	Nonlinear differential equation	No	Forms of deltas
1961	Le Mehaute, Brebner	$\sin\frac{7\alpha}{4}$			Diffusion equation closed-form solution	No	Groins-sudden dump sinusoidol undulation, equilibrium shape between groins
1964	Grijm	$\sin 2\alpha$ implied	Small and large angle	No	Cylindrical system of coordinates-numerical or graphical method	No	Forms of deltas
1964	Bakker, Edelman	$K_1 \tan\alpha$ $K_2 \tan^{-1}\alpha$ $\frac{1}{K_1} = K_2 = 1.23$	Small angle $o<\tan\alpha<1.23$ large angle $1.23<\tan\alpha<\infty$	No	Nonlinear differential equations, closed-form solutions	No	Forms of deltas
1968	Bakker	$\tan\alpha$	Very small angle	Yes (two-line theory)	System of linear differential equations 1) power series solution 2) closed-form solution 3) closed-form	No	Groins and combinations of groins 1) simple groins 2) stationary shorelines 3) sand-wave propagation
1970	Bakker, Breteler, Roos,	$\sin 2\alpha$	Small angle		Numerical method	No	Groins
1972	Price, Tomlinson, Willis	$\sin 2\alpha$ $\alpha = \alpha_o - \tan^{-1}\frac{\partial y}{\partial x}$	Small angle	No	Numerical method based on Pelnard-Considere (1956)	Laboratory with crushed coal	Groin
1972	Lepetit	$\sin\alpha \cos\alpha$	Small angle	No	Numerical method based on Pelnard-Considere (1956)	Laboratory with Bakelite	Groin (updrift and downdrift)
1972	Sylvester, Ho				Empirical fit	Yes	Crenulated-shaped bay or spiral beaches
1972	Leblond	Proportional to longshore current (radiation-stress)	Small angle	No	Numerical method		Spiral beaches
1973	Komar	$\sin 2\alpha$	Small angle	No	Numerical method	No	Growth of deltas, reorientation of beaches between two headlands
1975	Komar	$\sin 2\alpha$	Any angle		Numerical method based on empirical model refraction-diffraction		Hooked beaches (spiral beaches)
1976	Hulsbergen, Van Bochove, Bakker	$\sin 2\alpha$	Small angle	Yes	Application of the two-line theory of Bakker	Laboratory with dune sand	Groin
1976	Weggel	$\sin 2\alpha$	Small angle	Yes	Mathematical and numerical formulation	No	Groin

Table 2. Summary of mathematical models for shoreline evolution.--continued

Variation of beach slope taken into account	Effect of Diffraction	Modification by refraction	Variable wave direction	Variation of sea level	Effect of rip currents	Application to prototype cases	Main Conclusions
No (one-line theory)	No	No	No	No	No	No	First significant milestone of introduction of mathematical modeling to the study of shoreline evolution
No	No	No	No	No	No	No	Extension of the method of Pelnard-Considere to other idealized cases
No	No	No	No	No	No	No	Two forms of solution: One with concave shoreline (small angle) One with convex shoreline (large angle)
No	No	No	No	No	No	No	Same as Larras (1957)
No	No	No	No	No	No	No	Two forms of solutions as in 1960 applied to a number of deltas, idealized cases
No	No	No	No	No	No	No	Instability of shoreline under large angle (even a straight shoreline)
Implicitly (through the two-line theory)	No	No	No	No	No	No	The most significant contribution since 1956 demonstrating the influence of beach slope
Implicitly	Yes (periodic wave) (constant depth)	No	Yes	No	No	No	Influence of diffraction changing wave condition leading to stable shoreline near groin
No	Yes, but not completely formulated, not applied	No	No	No	No	No	Experimental verification for more diffracting waves, i.e., updrift, fairly satisfactory
No	No	No	No	No	No	No	Good experimental verification
	Combined effect, refraction-diffraction	Yes	No	No		Fit with prototype cases	Combined effect of refraction & diffraction in protected areas; affected also to geological time evolution
No	Yes, but unsuccessfully (numerical instability)	Yes	No	No	No	No	Qualitative fit only, unsuccessful; two large distances between data point. Complexity of combined refraction-diffraction effect.
No	No	No	No	No	No	No	Numerical application of Pelnard-Considere
	Combined refraction and diffraction						Empirical development
Implicitly	No	No	No	No	No	No	Fairly good experimental verification except near groin
Yes (in principle)	Yes (in principle)	No	Yes (statistically)	Yes	No	No	Aimed at investigating real cases on Great Lakes

superimposed on the long-term evolution for which an analysis can be done independently.

Among the significant recent reports leading toward understanding of surf zone circulation and related bottom topography are: Bowen and Inman (1969) who advocate the presence of edge waves as a cause of rip currents and beach cusps; Hino (1974) who states that rip currents are the result of mobility of the sand bed and hydrodynamic instability; Sonu (1972) and Noda (1972) demonstrated that a perturbation on bottom topography causing waves to refract and have varying intensity along the shore induces a variation in radiation stress which in turn enhances rip currents; finally, Liu and Mei (1976) applied the radiation-stress concept to a groin perpendicular to shore and to an offshore breakwater.

These investigations offer at least partial answers to a number of important problems, important in understanding shoreline processes. It definitely indicates that the radiation-stress approach holds the potential key to understanding many types of nearshore currents, heretofore unexplored. It is also evident that the study of surf zone hydrodynamics will rapidly reach a plateau if sand-water interaction problems are not mastered, and at this stage, these can only be considered empirically. Determinism leaves off with the inception of turbulence.

Even though the dynamics of nearshore currents hold the key to understanding of beach processes, application of the methodology based on radiation stress to investigate shoreline evolution mathematically is still beyond the state-of-the-art.

Both approaches could be pursued in parallel and the results of the scientific approach could slowly be incorporated into a practical engineering model.

Conclusions based on the literature survey, as summarized in Table 2, are:

a. There is sufficient laboratory verification to give credibility to a mathematical approach to the study of shoreline evolution for small angles of wave approach.

b. For large angles of incidence, there is a lesser chance at arriving at a successful formulation since shorelines are then unstable and the resulting shoreline evolution could not be predicted without the initiation of more basic research beyond the present state of knowledge.

c. Even though no field measurements subsequent to mathematical predictions have been found in the literature, many practicing engineers have applied the theory of Pelnard-Considere (1956) to predict shore evolution by taking into account variable wave climate. The method is easy to apply and provides valuable information.

d. Engineering applications to prototype cases based on more sophisticated approaches such as given by the two-line theory of Bakker (1968b) are not known. These more sophisticated approaches can be currently considered as belonging to the realm of research rather than of engineering practice.

e. Local effects, diffraction, rip currents, wave refraction and interaction between these effects are, at present, still not so conveniently formulated to be used by practicing engineers. Introduction of these effects, if and when important in the mathematical formulation, is feasible but will require further investigation.

f. A simple numerical scheme that can be used by design engineers and planners and which includes theoretical or empirically all important effects could be developed. Effects that should be included in the mathematical model are wave diffraction, loss of sand by rip currents along groins, sea (lake) level variation, and beach slope variation near groins.

g. The introduction of the concept of radiation stress in the mathematical formulation is not recommended at this time, but research related to this approach should be pursued in view of the eventual input that subsequent results could have on then existing operational mathematical models.

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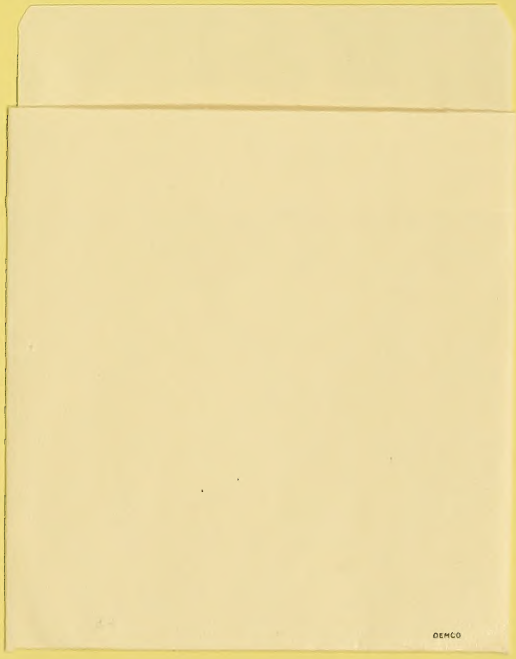
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